

# Multivariable Al-Salam & Carlitz polynomials associated with the type $A$ $q$ -Dunkl kernel

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The Al-Salam & Carlitz polynomials are  $q$ -generalizations of the classical Hermite polynomials. Multivariable generalizations of these polynomials are introduced via a generating function involving a multivariable hypergeometric function which is the  $q$ -analogue of the type- $A$  Dunkl integral kernel. An eigenoperator is established for these polynomials and this is used to prove orthogonality with respect to a certain Jackson integral inner product. This inner product is normalized by deriving a  $q$ -analogue of the Mehta integral, and the corresponding normalization of the multivariable Al-Salam & Carlitz polynomials is derived from a Pieri-type formula. Various other special properties of the polynomials are also presented, including their relationship to the shifted Macdonald polynomials and the big- $q$  Jacobi polynomials.

## 1 Introduction

The advent of Macdonald polynomials [21] has opened the way to a theory of hypergeometric functions in which the Macdonald polynomials are the underlying basis. Progress in this direction has been made by Macdonald [19], Kaneko [15] and Lassalle [17]. Present in the work of Macdonald and Lassalle is the hypergeometric function

$${}_0\mathcal{F}_0(x; y; q, t) := \sum_{\kappa} \frac{t^{b(\kappa)}}{h'_{\kappa}(q, t) P(1, t, \dots, t^{n-1}; q, t)} P_{\kappa}(x; q, t) P_{\kappa}(y; q, t). \quad (1.1)$$

where  $x$  (similarly  $y$ ) denotes the  $n$ -tuple  $(x_1, \dots, x_n)$ , the sum is over all partitions of length  $n$  and  $P_{\kappa}$  denotes the Macdonald polynomial normalized so that the coefficient of the leading monomial is unity, while Kaneko defines the hypergeometric function

$${}_0\psi_0(x; y; q, t) := \sum_{\kappa} \frac{(-1)^{|\kappa|} q^{b(\kappa')}}{h'_{\kappa}(q, t) P(1, t, \dots, t^{n-1}; q, t)} P_{\kappa}(x; q, t) P_{\kappa}(y; q, t). \quad (1.2)$$

The quantities  $h'_{\kappa}(q, t)$  and  $b(\kappa)$  in (1.1) and (1.2) are defined by

$$h'_{\kappa}(q, t) := \prod_{(i,j) \in \kappa} (1 - q^{\kappa_i - j + 1} t^{\kappa'_j - i}), \quad b(\kappa) = \sum_{j=1}^n (j-1) \kappa_j \quad (1.3)$$

and  $\kappa'$  in (1.2) denotes the partition conjugate to  $\kappa$ .

Using (1.3), the explicit formulas [21]

$$t^{-b(\kappa)} P(1, t, \dots, t^{n-1}; q, t) = \frac{\prod_{(i,j) \in \kappa} (1 - q^{j-1} t^{n-i+1})}{h_{\kappa}(q, t)}, \quad h_{\kappa}(q, t) = \prod_{(i,j) \in \kappa} (1 - q^{\kappa_i - j} t^{\kappa'_j - i + 1}) \quad (1.4)$$

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and the facts that  $P_\kappa(x; q, t) = P_\kappa(x; q^{-1}, t^{-1})$ ,  $c^{|\kappa|} P_\kappa(x; q, t) = P_\kappa(cx; q, t)$ , we see that the hypergeometric functions (1.1) and (1.2) are related by

$${}_0\mathcal{F}_0(x; y; q^{-1}, t^{-1}) = {}_0\psi_0(x; t^{n-1}qy; q, t) \quad (1.5)$$

Furthermore, with  $t = q^{1/\alpha}$ , in the limit  $q \rightarrow 1$  (1.1) with  $y$  replaced by  $(1 - q)y$  and (1.2) with  $y$  replaced by  $-(1 - q)y$  become equal:

$$\lim_{q \rightarrow 1} {}_0\mathcal{F}_0(x; (1 - q)y; q, q^{1/\alpha}) = \lim_{q \rightarrow 1} {}_0\psi_0(x; -(1 - q)y; q, t) = {}_0\mathcal{F}_0^{(\alpha)}(x; y) \quad (1.6)$$

where

$${}_0\mathcal{F}_0^{(\alpha)}(x; y) := \sum_{\kappa} \frac{\alpha^{|\kappa|} P_{\kappa}^{(\alpha)}(x) P_{\kappa}^{(\alpha)}(y)}{d'_{\kappa} P_{\kappa}^{(\alpha)}(1^n)} \quad (1.7)$$

and the limit is to be taken term-wise in the series. In (1.7),  $P_{\kappa}^{(\alpha)}(x)$  denotes the Jack polynomial normalized so that the coefficient of the leading monomial is unity and

$$d'_{\kappa} = \prod_{(i,j) \in \kappa} \left( \alpha(\kappa_i - j + 1) + (\kappa'_j - i) \right).$$

The hypergeometric function  ${}_0\mathcal{F}_0(x; y)$  has a number of noteworthy features. It was first observed by Lassalle (see [4] for a published version of most of the results of this work) that this function is the explicit realization of the symmetric type  $A$  integral kernel occurring in the work of Dunkl [9, 10]. As such it occurs in multidimensional generalizations of the Laplace, Hankel and Fourier transforms, as well as in the study of the type  $A$  Calogero-Sutherland quantum many body system (see e.g. [4]). Moreover, associated with this kernel are a class of multivariable polynomials generalizing the classical Hermite polynomials [19, 18, 4, 8, 23]. It is the objective of this article to investigate the  $q$ -analogues of these polynomials associated with  ${}_0\mathcal{F}_0(x; y; q, t)$  and  ${}_0\psi_0(x; y; q, t)$ .

After reviewing some of the salient properties of the single-variable Al-Salam&Carlitz polynomials  $U_n^{(a)}$ , Section 2 contains the definition of the multivariable Al-Salam&Carlitz polynomials  $U_{\kappa}^{(a)}$  in terms of a generating function involving  ${}_0\mathcal{F}_0(x; y; q, t)$ . In Section 3 these polynomials are related to the Macdonald polynomials by a  $q$ -exponential operator formula, and they are also shown to satisfy an eigenvalue equation. Three different forms of the eigenoperator are given, one of which is manifestly Hermitian with respect to a particular Jackson integral inner product, thus establishing an orthogonality relation.

In Section 4 special properties of the multivariable Al-Salam&Carlitz polynomials are established. One of these properties is a Pieri-type formula expressing  $e_1 U_{\kappa}^{(a)}$ , where  $e_1$  denotes the first elementary symmetric function, in terms of a linear combination of  $U_{\lambda}^{(a)}$ . This formula is used to calculate the normalization of  $U_{\kappa}^{(a)}$  with respect to the Jackson integral inner product, and the normalization in turn is used in the derivation of some integral representations. Section 5 contains a discussion on the relationship between the multivariable Al-Salam&Carlitz polynomials and the big  $q$ -Jacobi polynomials introduced recently by Stokman [26]. We conclude with two Appendices. The first is an expansion formula required in calculating the normalization of the inner product, while in the second a determinant formula for the  $U_{\kappa}^{(a)}$  in terms of the  $U_n^{(a)}$  is given in the special case  $t = q$ .

## 2 Generalization of the polynomials of Al-Salam and Carlitz

## 2.1 The case $n = 1$

For  $n = 1$  we have  ${}_0\mathcal{F}_0^{(\alpha)}(x; y) = e^{xy}$ . Replacing  $y$  by  $2y$  and multiplying by  $e^{-x^2}$  gives the generating function for the classical Hermite polynomials:

$$e^{-x^2} e^{2xy} = \sum_{n=0}^{\infty} \frac{H_n(y)x^n}{n!}. \quad (2.1)$$

We recall that the classical Hermite polynomials are eigenfunctions of the differential operator

$$\frac{d^2}{dy^2} - 2y \frac{d}{dy} \quad (2.2)$$

and have the orthogonality

$$\int_{-\infty}^{\infty} dy e^{-y^2} H_m(y) H_n(y) = \sqrt{\pi} 2^n n! \delta_{m,n}. \quad (2.3)$$

In the  $q$ -case for  $n = 1$

$${}_0\mathcal{F}_0^{(\alpha)}(x; y; q, t) = e_q(xy) := \sum_{n=0}^{\infty} \frac{(xy)^n}{(q; q)_n} = \frac{1}{(xy; q)_{\infty}} \quad (2.4)$$

$${}_0\psi_0(x; y; q, t) = E_q(-xy) := \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n-1)/2} (xy)^n}{(q; q)_n} = (xy; q)_{\infty}, \quad (2.5)$$

(( $x; q)_n$ ,  $(x; q)_{\infty}$  have their usual meaning) where the final equalities are valid for  $|q| < 1$ . The  $q$ -generalizations of (2.1) associated with (2.4) and (2.5) have been given by Al-Salam and Carlitz [1].

Consider first (2.4). Al-Salam and Carlitz defined a family of polynomials  $\{U_n^{(a)}(y; q)\}$ , ( $a < 0$ ) by the generating function

$$\rho_a(x; q) e_q(xy) = \sum_{n=0}^{\infty} U_n^{(a)}(y; q) \frac{x^n}{(q; q)_n}, \quad \rho_a(x; q) := (x; q)_{\infty} (ax; q)_{\infty} = E_q(-x) E_q(-ax) \quad (2.6)$$

and established the orthogonality

$$\int_a^1 w_U^{(a)}(x; q) U_m^{(a)}(x; q) U_n^{(a)}(x; q) d_q x = (1 - q) (-a)^n q^{n(n-1)/2} (q; q)_n \delta_{m,n}, \quad (2.7)$$

where

$$w_U^{(a)}(x; q) := \frac{(qx; q)_{\infty} (\frac{qx}{a}; q)_{\infty}}{(q; q)_{\infty} (a; q)_{\infty} (\frac{q}{a}; q)_{\infty}} \quad (2.8)$$

$$\int_a^1 f(x) d_q x := (1 - q) \left( \sum_{n=0}^{\infty} f(q^n) q^n - a \sum_{n=0}^{\infty} f(aq^n) q^n \right), \quad (a < 0) \quad (2.9)$$

In the case  $a = -1$  the polynomials  $U_n^{(a)}(x; q)$  are referred to as the discrete  $q$ -Hermite polynomials [12], and they reduce to the classical Hermite polynomials in the  $q \rightarrow 1$  limit:

$$\lim_{q \rightarrow 1} (1 - q)^{-n/2} U_n^{(-1)}(x(1 - q)^{1/2}; q) = 2^{-n/2} H_n\left(\frac{x}{\sqrt{2}}\right). \quad (2.10)$$

For general  $a$ , consideration of the three term recurrence for  $U_n^{(a)}$  (eq. (4.3) below) shows

$$\lim_{q \rightarrow 1} (1 - q)^{-n/2} U_n^{(-q^r)}(x(1 - q)^{1/2}; q) = 2^{-n/2} H_n\left(\frac{x - r}{\sqrt{2}}\right). \quad (2.11)$$

In relation to (2.5), Al-Salam and Carlitz introduced the polynomials  $\{V_n^{(a)}(x; q)\}$  via the generating function

$$\frac{1}{\rho_a(x; q)} E_q(-xy) = \sum_{n=0}^{\infty} V_n^{(a)}(y; q) \frac{(-1)^n q^{n(n-1)/2} x^n}{(q; q)_n}. \quad (2.12)$$

Note from (2.6) and (2.4) that

$$\frac{1}{\rho_a(x; q)} = \frac{1}{(x; q)_{\infty} (ax; q)_{\infty}} = e_q(x) e_q(ax). \quad (2.13)$$

These polynomials were shown to satisfy the orthogonality

$$\int_1^{\infty} w_V(x; q) V_m^{(a)}(x; q) V_n^{(a)}(x; q) d_q x = (1 - q) a^m (q; q)_m q^{-m^2} \delta_{m,n} \quad (2.14)$$

where

$$w_V(x; q) = \frac{(q; q)_{\infty} (\frac{1}{a}; q)_{\infty} (qa; q)_{\infty}}{(x; q)'_{\infty} (\frac{x}{a}; q)_{\infty}} \quad (2.15)$$

(the dash in  $(x; q)'_{\infty}$  denotes that any factor which vanishes for a particular choice of  $x$  is to be deleted) with

$$\int_1^{\infty} f(x) d_q x := (1 - q) \sum_{n=0}^{\infty} f(q^{-n}) q^{-n} \quad (2.16)$$

Furthermore it was established in [1] that the polynomials  $\{U_n^{(a)}(x; q)\}$  and  $\{V_n^{(a)}(x; q)\}$  are simply related:

$$V_n^{(a)}(x; q) = U_n^{(a)}(x; q^{-1}). \quad (2.17)$$

This can be seen by first noting from (2.4), (2.5) and (1.5) that

$$E_{q^{-1}}(-x) = e_q(qx) \quad (2.18)$$

and using this in (2.12) with  $q$  replaced by  $q^{-1}$  to obtain the generating function

$$\rho_a(qx; q) e_q(qxy) = \sum_{n=0}^{\infty} V_n^{(a)}(y; q^{-1}) \frac{(qx)^n}{(q; q)_n} \quad (2.19)$$

Comparison with (2.6) implies (2.17).

Using (2.17) we can check that the orthogonality (2.14) is equivalent to (2.7) with  $q$  replaced by  $q^{-1}$ . This is done by first generalizing  $w_U^{(a)}$  to include an auxiliary parameter  $\mu$ :

$$w_U^{(a)}(x; q, \mu) := \frac{E_q(-\mu qx) E_q(-\mu \frac{qx}{a})}{E_q(-\mu q) E_q(-a) E_q(-\mu \frac{q}{a})} \quad (2.20)$$

With this definition (2.18) gives

$$w_U^{(a)}(x; q^{-1}, \mu) = \frac{e_q(\mu x) e_q(\mu \frac{x}{a})}{e_q(\mu) e_q(qa) e_q(\frac{\mu}{a})}. \quad (2.21)$$

Now, comparing (2.1) and (2.8), we see from (2.9) that

$$\frac{1}{1 - q} \int_a^1 w_U^{(a)}(x; q) f(x) d_q x := \sum_{n=0}^{\infty} \lim_{\mu \rightarrow 1} w_U^{(a)}(q^n; q, \mu) f(q^n) q^n - a \sum_{n=0}^{\infty} \lim_{\mu \rightarrow 1} w_U^{(a)}(aq^n; q, \mu) f(aq^n) q^n. \quad (2.22)$$

Next we set  $a = -q^p$  for some fixed  $p \in \mathbb{Z}^+$ , and replace  $q$  by  $q^{-1}$  in (2.22). From (2.21) and the product formula in (2.4), for  $x = -aq^n$  ( $n = 0, 1, \dots$ ),  $\lim_{\mu \rightarrow 1} w_U^{(a)}(x; q^{-1}, \mu) = 0$  while for  $x = q^n$ ,  $n = 0, 1, \dots$ ,

$$\lim_{\mu \rightarrow 1} w_U^{(a)}(x; q^{-1}, \mu) = w_V^{(a)}(x; q).$$

Substituting in (2.22) gives

$$\frac{1}{1-q} \int_a^1 w_U^{(a)}(x; q) f(x) d_q x \Big|_{q \mapsto q^{-1}} = \sum_{n=0}^{\infty} w_V^{(a)}(q^{-n}; q^{-1}) f(q^{-n}) q^{-n}, \quad (2.23)$$

which is the required inter-relationship. A simple lemma of Stembridge [25] show that this formula, proved valid for  $a = -q^p$ , must remain true for all  $a$  such that both sides are defined.

## 2.2 Generating function for general $n$

For general  $n$  the generating function for the classical Hermite polynomials (2.1) has the generalization [18, 4]

$$e^{-x_1^2} \dots e^{-x_n^2} {}_0\mathcal{F}_0^{(\alpha)}(x; 2y) = \sum_{\kappa} \frac{\alpha^{|\kappa|} H_{\kappa}(y; \alpha) P_{\kappa}^{(\alpha)}(x)}{d'_{\kappa}}, \quad (2.24)$$

where the polynomials  $\{H_{\kappa}(y; \alpha)\}$  are referred to as the generalized Hermite polynomials. These polynomials have an expansion in terms of  $\{P_{\sigma}^{(\alpha)}(y)\}$  of the form

$$H_{\kappa}(y; \alpha) = \frac{2^{|\kappa|}}{P_{\kappa}^{(\alpha)}(1^n)} P_{\kappa}^{(\alpha)}(y) + \sum_{|\mu| < |\kappa|} c_{\kappa\mu} P_{\mu}^{(\alpha)}(y). \quad (2.25)$$

With

$$D_0 := \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} + \frac{2}{\alpha} \sum_{i \neq j} \frac{1}{x_i - x_j} \frac{\partial}{\partial x_i}, \quad E_1 := \sum_{i=1}^n x_i \frac{\partial}{\partial x_i}, \quad (2.26)$$

the generalized Hermite polynomials are also related to the Jack polynomials via the exponential operator formula [18, 4, eq.(3.21)]

$$\frac{2^{|\kappa|}}{P_{\kappa}^{(\alpha)}(1^n)} \exp\left(-\frac{1}{4} D_0\right) P_{\kappa}^{(\alpha)}(x) = H_{\kappa}(x; \alpha) \quad (2.27)$$

and they are eigenfunctions of the operator

$$\tilde{H}^{(H)} := D_0 - 2E_1. \quad (2.28)$$

They also satisfy the orthogonality

$$\int_{(-\infty, \infty)^n} H_{\kappa}(y; \alpha) H_{\sigma}(y; \alpha) d\mu^{(H)}(y) = \frac{2^{|\kappa|} d'_{\kappa} \mathcal{N}_0^{(H)}}{P_{\kappa}^{(\alpha)}(1^n)} \delta_{\kappa, \sigma} \quad (2.29)$$

where

$$d\mu^{(H)}(y) := \prod_{j=1}^n e^{-y_j^2} \prod_{1 \leq j < k \leq n} |y_j - y_k|^{2/\alpha} dy_1 \dots dy_n, \quad (2.30)$$

$$\mathcal{N}_0^{(H)} := \int_{(-\infty, \infty)^n} d\mu^{(H)}(y) = 2^{-n(n-1)/2} \pi^{n/2} \prod_{j=0}^{n-1} \frac{\Gamma(1 + (j+1)/\alpha)}{\Gamma(1 + 1/\alpha)}. \quad (2.31)$$

In light of this result, the single variable polynomials of Al-Salam and Carlitz suggest an  $n$ -dimensional generalization. Thus we introduce two sets of multivariable polynomials  $\{U_\kappa^{(a)}(y; q, t)\}$  and  $\{V_\kappa^{(a)}(y; q, t)\}$  by the generating functions

$$\rho_a(x_1; q) \cdots \rho_a(x_n; q) {}_0\mathcal{F}_0(x; y; q, t) = \sum_\kappa \frac{t^{b(\kappa)} U_\kappa^{(a)}(y; q, t) P_\kappa(x; q, t)}{h'_\kappa(q, t) P_\kappa(1, t, \dots, t^{n-1}; q, t)} \quad (2.32)$$

and

$$\frac{1}{\rho_a(t^{-(n-1)}x_1; q) \cdots \rho_a(t^{-(n-1)}x_n; q)} {}_0\psi_0(x; y; q, t) = \sum_\kappa \frac{(-1)^{|\kappa|} q^{b(\kappa')} V_\kappa^{(a)}(y; q, t) P_\kappa(x; q, t)}{h'_\kappa(q, t) P_\kappa(1, t, \dots, t^{n-1}; q, t)}. \quad (2.33)$$

With these definitions the polynomials  $U_\kappa^{(a)}$  and  $V_\kappa^{(a)}$  are simply related. This is seen by replacing  $q, t$  by  $q^{-1}, t^{-1}$  in (2.32). Use of (2.18) and (1.5) and comparison with (2.33) shows

$$V_\kappa^{(a)}(y; q, t) = U_\kappa^{(a)}(y; q^{-1}, t^{-1}) \quad (2.34)$$

in analogy with (2.17). Also, it follows from the generating function (2.32) that the polynomials  $U_\kappa^{(a)}$  have an expansion in terms of the Macdonald polynomials of the form

$$U_\kappa^{(a)}(x; q, t) = P_\kappa(x; q, t) + \sum_{|\nu| < |\kappa|} a_{\kappa\nu} P_\nu(x; q, t) \quad (2.35)$$

in analogy with (2.25).

The generalized Hermite polynomials can be reclaimed from the polynomials  $U_\kappa^{(-1)}$  by a limiting procedure analogous to (2.10). To see this put  $q = t^\alpha$  and  $a = -1$  in (2.32), and note from (2.6) and (2.5) that  $\rho_1(x; q) = (x^2; q^2)_\infty$ . Then replace  $x$  by  $(1-q)^{1/2}x$  and  $y$  by  $(1-q)^{1/2}y$ , and consider the limit  $q \rightarrow 1$ . Now  ${}_0\mathcal{F}_0((1-q)^{1/2}x; (1-q)^{1/2}y; q, t) = {}_0\mathcal{F}_0(x; (1-q)y; q, t)$  so we can take the limit in this term by using (1.6), while it follows from (2.5) that  $(x^2(1-q); q^2)_\infty \rightarrow e^{-x^2/2}$  as  $q \rightarrow 1$ . On the r.h.s. of (2.32) we note that in this limit

$$\frac{P_\kappa((1-q)^{1/2}x; q, q^{1/\alpha})}{P_\kappa(1, t, \dots, t^{n-1}; q, q^{1/\alpha})} = (1-q)^{|\kappa|/2} \frac{P_\kappa(x; q, q^{1/\alpha})}{P_\kappa(1, t, \dots, t^{n-1}; q, q^{1/\alpha})} \sim (1-q)^{|\kappa|/2} \frac{P_\kappa^{(\alpha)}(x)}{P_\kappa^{(\alpha)}(1^n)},$$

where  $P_\kappa^{(\alpha)}(x)$  denotes the Jack polynomial, while  $(1-q)^{|\kappa|/2} h'_\kappa(q, q^{1/\alpha}) \rightarrow d'_\kappa / \alpha^{|\kappa|}$ . Thus we have

$$e^{-x_1^2/2} \cdots e^{-x_n^2/2} {}_0\mathcal{F}_0^{(\alpha)}(x; y) = \sum_\kappa \frac{\alpha^{|\kappa|} \lim_{q \rightarrow 1} (1-q)^{-|\kappa|/2} U_\kappa^{(-1)}((1-q)^{1/2}y; q, q^{1/\alpha}) P_\kappa^{(\alpha)}(x)}{d'_\kappa P_\kappa^{(\alpha)}(1^n)}.$$

Comparison with (2.24) gives the multidimensional generalization of (2.10),

$$\lim_{q \rightarrow 1} (1-q)^{-|\kappa|/2} U_\kappa^{(-1)}((1-q)^{1/2}x; q, q^{1/\alpha}) = 2^{-|\kappa|/2} P_\kappa^{(\alpha)}(1^n) H_\kappa\left(\frac{x}{\sqrt{2}}; \alpha\right). \quad (2.36)$$

### 3 Exponential operator formula, eigenvalue equation and Hermiticity

To further develop the theory of the polynomials  $\{U_\kappa^{(a)}(x; q, t)\}$ , we must revise the definitions of operators  $Y_i^{\pm 1}$  and  $D_i$ , which play a role in the theory of non-symmetric Macdonald polynomials.

These operators in turn are defined in terms of certain operators  $T_i$ ,  $\omega$  appearing in the theory of type  $A$  affine Hecke algebras. The Demazure-Lustig operators  $T_i$  are defined by

$$T_i := t + \frac{tx_i - x_{i+1}}{x_i - x_{i+1}}(s_i - 1) \quad i = 1, \dots, n-1 \quad (3.1)$$

while

$$\omega := s_{n-1} \cdots s_2 s_1 \tau_1 = s_{n-1} \cdots s_i \tau_i s_{i-1} \cdots s_1. \quad (3.2)$$

For future reference we note that the operators  $T_i$  and  $\omega$  have the properties

$$\begin{aligned} T_i^{-1} x_{i+1} &= t^{-1} x_i T_i & T_i^{-1} x_i &= x_{i+1} T_i^{-1} + (t^{-1} - 1)x_i \\ T_i x_i &= tx_{i+1} T_i^{-1} & T_i x_{i+1} &= x_i T_i + (t - 1)x_{i+1} \\ \omega x_i &= qx_n \omega & \omega x_{i+1} &= x_i \omega \end{aligned} \quad (3.3)$$

valid for  $1 \leq i \leq n-1$ .

Using the definitions (3.1) and (3.2) the operators  $Y_i$  [7, 20] are defined by

$$Y_i = t^{-n+i} T_i \cdots T_{n-1} \omega T_1^{-1} \cdots T_{i-1}^{-1}, \quad 1 \leq i \leq n. \quad (3.4)$$

They mutually commute and have the following relations with the operators  $T_i$ ,

$$T_i Y_{i+1} T_i = t Y_i \quad [T_i, Y_j] = 0, \quad j \neq i, i+1 \quad (3.5)$$

The  $q$ -analogue of the type  $A$  Dunkl operators were defined in [6] by means of

$$D_i := x_i^{-1} \left( 1 - t^{n-1} \left[ 1 + (t^{-1} - 1) \sum_{j=i+1}^n t^{j-i} T_{ij}^{-1} \right] Y_i \right) \quad (3.6)$$

where for  $i < j$ ,

$$\begin{aligned} T_{ij}^{-1} &:= T_i^{-1} T_{i+1}^{-1} \cdots T_{j-2}^{-1} T_{j-1}^{-1} T_{j-2}^{-1} \cdots T_i^{-1} \\ &= T_{j-1}^{-1} T_{j-2}^{-1} \cdots T_{i+1}^{-1} T_i^{-1} T_{i+1}^{-1} \cdots T_{j-1}^{-1} \end{aligned} \quad (3.7)$$

### 3.1 $q$ -exponential operator formula

From Theorem 5.2(c) and Proposition 5.4 of ref. [6], it follows that for any symmetric function  $f$  analytic in the neighbourhood of the origin,

$$f(D_1^{(x)}, \dots, D_n^{(x)})_0 \mathcal{F}_0(x; y; q, t) = f(y_1, \dots, y_n)_0 \mathcal{F}_0(x; y; q, t) \quad (3.8)$$

where the  $D_i$  are the operators (3.6) acting on the  $x$ -variables only. By choosing  $f(x_1, \dots, x_n) = \rho_a(x_1; q) \cdots \rho_a(x_n; q)$  and using the generating function (2.32) we see immediately that

$$\rho_a(D_1; q) \cdots \rho_a(D_n; q) P_\kappa(x; q, t) = U_\kappa^{(a)}(x; q, t), \quad (3.9)$$

which is the  $q$ -generalization of the exponential operator formula (2.27). Replacing  $q, t$  by  $q^{-1}, t^{-1}$  in (3.9) and using (2.34) and (2.18) gives

$$\frac{1}{\rho_a(q\widetilde{D}_1; q) \cdots \rho_a(q\widetilde{D}_n; q)} P_\kappa(x; q, t) = V_\kappa^{(a)}(x; q, t). \quad (3.10)$$

where  $\widetilde{D}_i$  denotes the operators (3.6) with  $q, t$  replaced by  $q^{-1}, t^{-1}$ . We remark that since the  $D_i$  and  $\widetilde{D}_i$  are degree lowering operators, only a finite number of terms in the power series expansion of the  $q$ -exponential operators give a non-zero contribution to (3.9) and (3.10).

### 3.2 First form of eigenoperator

With the polynomials  $\{U_\kappa^{(a)}(x; q, t)\}$  defined by the generating function (2.32), we want to derive the eigenoperator analogous to (2.28) which has  $\{U_\kappa^{(a)}(x; q, t)\}$  as eigenfunctions. To present our first representation of this eigenoperator, let  $\tau_i$  denote the  $q$ -shift operator of the  $i$ th variable,

$$\tau_i f(x_1, \dots, x_n) = f(x_1, \dots, x_{i-1}, qx_i, x_{i+1}, \dots, x_n), \quad (3.11)$$

let  $M_1$  denote the first Macdonald operator,

$$M_1 := \sum_{i=1}^n A_i(t) \tau_i, \quad A_i(t) := \prod_{\substack{p=1 \\ p \neq i}}^n \frac{tx_i - x_p}{x_i - x_p}, \quad (3.12)$$

put

$$E_k := \sum_{i=1}^n x^k A_i(t) \frac{\partial}{\partial_q x_i}, \quad \frac{\partial}{\partial_q x_i} := \frac{1 - \tau_i}{(1 - q)x_i} \quad (3.13)$$

and write  $\widetilde{M}_1$  for the operator with  $q$  and  $t$  replaced by  $q^{-1}$  and  $t^{-1}$ . The required eigenoperator can be written in terms of  $\widetilde{M}_1$  and  $E_0$  (c.f. the form of the eigenoperator for the generalized Hermite polynomials given in Proposition 3.2 of ref. [4]).

**Proposition 3.1** *We have*

$$\mathcal{H} U_\kappa^{(a)}(x; q, t) = \tilde{e}(\kappa) U_\kappa^{(a)}(x; q, t) \quad (3.14)$$

where

$$\mathcal{H} = \widetilde{M}_1 - (1 + a)[E_0, \widetilde{M}_1] + a[E_0, [E_0, \widetilde{M}_1]] \quad (3.15)$$

and  $\tilde{e}(\kappa) = \sum_{i=1}^n q^{-\kappa_i} t^{-n+i}$ .

**Proof.** We first calculate the action of the Macdonald operator  $\widetilde{M}_1^{(x)}$  (the superscript  $(x)$  denotes that this operator acts on the  $x$ -variables only) on the l.h.s. of the generating function (2.32). We have

$$\begin{aligned} & \widetilde{M}_1^{(x)} \prod_{i=1}^n \rho_a(x; q) {}_0\mathcal{F}_0(x; y; q, t) \\ &= \sum_{p=1}^n A_p^{(x)}(t^{-1}) \rho_a(x; q) (1 - q^{-1}x_p)(1 - q^{-1}ax_p) \tau_{p,x}^{-1} {}_0\mathcal{F}_0(x; y; q, t) \\ &= \prod_{i=1}^n \rho_a(x; q) \left( \widetilde{M}_1^{(x)} - q^{-1}(1 + a) \sum_{p=1}^n x_p A_p^{(x)}(t^{-1}) \tau_{p,x}^{-1} + aq^{-2} \sum_{p=1}^n x_p^2 \right. \\ & \quad \left. \times A_p^{(x)}(t^{-1}) \tau_{p,x}^{-1} \right) {}_0\mathcal{F}_0(x; y; q, t) \end{aligned} \quad (3.16)$$

Using the readily verified identities

$$[\widetilde{M}_1^{(x)}, p_1(x)] = (q^{-1} - 1) \sum_{p=1}^n x_p A_p^{(x)}(t^{-1}) \tau_{p,x}^{-1} \quad (3.17)$$

$$[[\widetilde{M}_1^{(x)}, p_1(x)], p_1(x)] = (q^{-1} - 1)^2 \sum_{p=1}^n x_p^2 A_p^{(x)}(t^{-1}) \tau_{p,x}^{-1} \quad (3.18)$$



where  $p_1(x) := \sum_{i=1}^n x_i$ , the sums over  $p$  in (3.16) can be replaced in favour of the commutators in (3.17) and (3.18). But from the fact that  $\widetilde{M}_1$  is an eigenoperator of  $\{P_\kappa(x; q, t)\}$  we see from the definition of  ${}_0\mathcal{F}_0$  that

$$\widetilde{M}_1^{(x)} {}_0\mathcal{F}_0(x; y; q, t) = \widetilde{M}_1^{(y)} {}_0\mathcal{F}_0(x; y; q, t). \quad (3.19)$$

Furthermore, from the work of Lassalle [17] we know that

$$(1 - q)E_0^{(y)} {}_0\mathcal{F}_0(x; y; q, t) = p_1(x) {}_0\mathcal{F}_0(x; y; q, t) \quad (3.20)$$

Hence (3.16) can be rewritten to read

$$\begin{aligned} \widetilde{M}_1^{(x)} \prod_{i=1}^n \rho_a(x_i; q) {}_0\mathcal{F}_0(x; y; q, t) &= \left( \widetilde{M}_1^{(y)} - (1 + a)[E_0^{(y)}, \widetilde{M}_1^{(y)}] + a[E_0^{(y)}, [E_0^{(y)}, \widetilde{M}_1^{(y)}]] \right) \\ &\quad \times \prod_{i=1}^n \rho_a(x_i; q) {}_0\mathcal{F}_0(x; y; q, t) \end{aligned} \quad (3.21)$$

Substituting the r.h.s. of the generating function (2.32) for  $\prod_{i=1}^n \rho_a(x_i; q) {}_0\mathcal{F}_0(x; y; q, t)$  allows the action of the operator to be computed on the l.h.s. and implies the stated eigenvalue equation.  $\square$

### 3.3 Second form of eigenoperator

It is instructive to compare the eigenvalue equation (3.14) with eigenoperator (3.15) to the one satisfied by the one variable Al-Salam&Carlitz polynomials  $U_n^{(a)}$ . Let  $\tau$  denote the  $q$ -shift operator in one variable, and  $D := x^{-1}(1 - \tau)$ . Then

$$\left(1 - (1 + a)D + aD^2\right) \tau^{-1} U_n^{(a)} = q^{-n} U_n^{(a)} \quad (3.22)$$

with a similar equation for  $V_n(x; q)$ , with  $q \rightarrow q^{-1}$  (eq. (3.22) is equivalent to the hypergeometric-type difference equation given on pg. 125 of Askey and Suslov [3]). This is of a different form to that implied by (3.15) with  $n = 1$ . A multivariable analogue of (3.22) exists, but first some preliminary results are needed.

**Lemma 3.2** *When acting on the space of symmetric functions*

$$E_0 = \frac{1}{1 - q} \sum_{i=1}^n D_i \quad \widetilde{M}_1 = t^{1-n} \sum_{i=1}^n Y_i^{-1}$$

**Proof.** The first identity was given in [6, Lemma 5.3]. The proof of the second identity follows a similar line of reasoning, which we give for completeness. Let

$$A_{i,m}(t^{-1}) := \prod_{\substack{j=1 \\ j \neq i}}^m \frac{t^{-1}x_i - x_j}{x_i - x_j} \quad \widetilde{M}_1^{(m)} := \sum_{i=1}^m A_{i,m}(t^{-1}) \tau_i^{-1}$$

We shall in fact prove the following stronger result by induction:

$$\widetilde{M}_1^{(m)} = t^{1-m} \sum_{i=1}^m Y_i^{-1}, \quad m = 1, \dots, n. \quad (3.23)$$

From the explicit representation (3.4) for  $Y_i$  (and hence  $Y_i^{-1}$ ), we have that acting on symmetric functions,  $Y_1^{-1} = \tau_1^{-1}$ , so (3.23) is true in the case  $m = 1$ . By induction, the identity (3.23) is equivalent to the identity

$$Y_m^{-1} = t^{m-1} \widetilde{M}_1^{(m)} - t^{m-2} \widetilde{M}_1^{(m-1)}$$

The action of the operators  $Y_m^{-1}$  on symmetric functions can be generated recursively from  $Y_m^{-1} = T_{m-1} Y_{m-1}^{-1}$  (which follows from (3.5)). It thus suffices to show that the operators

$$\begin{aligned} R_m &:= t^{m-1} \widetilde{M}_1^{(m)} - t^{m-2} \widetilde{M}_1^{(m-1)} \\ &= t^{m-1} A_{m,m}(t^{-1}) \tau_m^{-1} + t^{m-2} (1-t) \sum_{i=1}^{m-1} \frac{x_m}{x_i - x_m} A_{i,m-1}(t^{-1}) \tau_i^{-1} \end{aligned}$$

obey the same relation:  $R_m = T_{m-1} R_{m-1}$ . This can be verified directly.  $\square$

Using the above lemma, we can now prove the following result

**Proposition 3.3** *Acting on the space of symmetric functions, we have*

$$[E_0, \widetilde{M}_1] = \sum_{i=1}^n t^{1-i} D_i Y_i^{-1} \quad (3.24)$$

$$[E_0, [E_0, \widetilde{M}_1]] = \sum_{i=1}^n t^{1-i} D_i^2 Y_i^{-1} + (1-t^{-1}) \sum_{1 \leq i < j \leq n} t^{1-i} D_j D_i Y_i^{-1} \quad (3.25)$$

**Proof.** We note from [6, Lemma 3.3] that

$$[Y_j^{-1}, D_i] = \begin{cases} t^{i-j}(1-t) T_{ij} D_j Y_j^{-1} & i < j \\ t^{j-i}(1-t) Y_j^{-1} Y_i T_{ji} D_i Y_j^{-1} & i > j \end{cases} \quad (3.26)$$

$$\begin{aligned} [Y_i^{-1}, D_i] &= (q-1) D_i Y_i^{-1} + (t-1) \sum_{p=i+1}^n t^{-p+i} Y_i^{-1} Y_p T_{ip} D_p Y_i^{-1} \\ &\quad + q(t-1) \sum_{p=1}^{i-1} t^{p-i} T_{ip} D_i Y_i^{-1} \end{aligned} \quad (3.27)$$

If we now consider the commutator  $[E_0, \widetilde{M}_1]$ , use Lemma 3.2 and then (3.26), (3.27), we find that

$$[E_0, \widetilde{M}_1] = \sum_{i=1}^n D_i Y_i^{-1} + (t-1) \sum_{1 \leq i < j \leq n} t^{i-j} T_{ij} D_j Y_j^{-1}$$

But for  $i < j$ ,  $T_{ij} D_j Y_j^{-1} = D_i Y_i^{-1} T_{ij}$ , which puts the operators  $T_{ij}$  on the right in the above expression and allows us to use the fact that when acting on symmetric functions,  $T_{ij} = t^{2(j-i)-1}$ . Simplification of the resulting expression yields the right hand side of (3.24). The proof of (3.25) proceeds likewise.  $\square$

The preceding result allows us to give an alternative form of the eigenoperator  $\mathcal{H}$  of the Al-Salam&Carlitz polynomials given in (3.15), namely

$$\mathcal{H} = t^{1-n} \sum_{i=1}^n Y_i^{-1} - (1+a) \sum_{i=1}^n t^{1-i} D_i Y_i^{-1} + a \sum_{i=1}^n t^{1-i} D_i^2 Y_i^{-1} + a(1-t^{-1}) \sum_{1 \leq i < j \leq n} t^{1-i} D_j D_i Y_i^{-1} \quad (3.28)$$

A comparison with (3.22) show that this correctly reproduces the one-variable result. We also point out that with this choice of  $\mathcal{H}$ ,  $a = -1$ ,  $q = t^\alpha$ , and  $x$  replaced by  $\sqrt{2(1-q)}x$ , in the limit  $q \rightarrow 1$  the eigenvalue equation (3.14) gives the eigenvalue equation for the generalized Hermite polynomials,

$$\tilde{H}^{(H)} H_\kappa(x; \alpha) = -2|\kappa| H_\kappa(x; \alpha),$$

where  $\tilde{H}^{(H)}$  is given by (2.28).

### 3.4 Hermiticity

In this section it will be shown that for  $t = q^k$ ,  $k \in \mathbb{Z}^+$ , the polynomials are orthogonal with respect to the inner product

$$\langle f|g \rangle^{(U)} := \int_{[a,1]^n} f(x)g(x) d_q \mu^{(U)}(x), \quad d_q \mu^{(U)}(x) := \Delta_q^{(k)}(x) \prod_{l=1}^n w_U(x_l; q) d_q x_l \quad (3.29)$$

where

$$\Delta_q^{(k)}(x_1, \dots, x_n) := \prod_{p=-(k-1)}^k \prod_{1 \leq i < j \leq n} (x_i - q^p x_j) \quad (3.30)$$

and  $f$  and  $g$  are assumed symmetric. This will be done by verifying that the eigenoperator  $\mathcal{H}$  in (3.14) is Hermitian with respect to (3.29). Actually we have not been able to deduce the Hermiticity of the Al-Salam&Carlitz eigenoperator  $\mathcal{H}$  directly from either (3.15) or (3.28). A third form is required, more amenable to contemplation of this feature.

First, we write

$$D_i = x_i^{-1} \left( 1 - t^{2n-i-1} I_{i,n-1}^{-1} Y_i \right) \quad I_{i,n-1}^{-1} := T_i^{-1} \cdots T_{n-1}^{-1} T_{n-1}^{-1} \cdots T_i^{-1}$$

Thus

$$\begin{aligned} \sum_{i=1}^n t^{1-i} D_i Y_i^{-1} &= \sum_{i=1}^n t^{1-i} \left( x_i^{-1} Y_i^{-1} - t^{2n-i-1} x_i^{-1} I_{i,n-1}^{-1} \right) \\ &= \sum_{i=1}^n t^{-n+i} T_{i-1}^{-1} \cdots T_1^{-1} x_1^{-1} \omega^{-1} T_{n-1} \cdots T_i \end{aligned} \quad (3.31)$$

and  $D_j D_i Y_i^{-1} = A_1^{(i,j)} + A_2^{(i,j)} + A_3^{(i,j)} + A_4^{(i,j)}$ , where

$$\begin{aligned} A_1^{(i,j)} &= x_j^{-1} x_i^{-1} Y_i^{-1} & A_2^{(i,j)} &= -t^{2n-j-1} x_j^{-1} I_{j,n-1}^{-1} Y_j x_i^{-1} Y_i^{-1} \\ A_3^{(i,j)} &= -t^{2n-i-1} x_j^{-1} x_i^{-1} I_{i,n-1}^{-1} & A_4^{(i,j)} &= t^{4n-i-j-2} x_j^{-1} I_{j,n-1}^{-1} Y_j x_i^{-1} I_{i,n-1}^{-1} \end{aligned}$$

Our aim is to write each of the terms  $A_p^{(i,j)}$  so that all occurrences of  $T_i^{\pm 1}$  are on the left of  $\omega^{\pm 1}$ , while shifting as many  $T_i^{\pm 1}$  as possible to the right (whence replacing them by  $t^{\pm 1}$  when acting on  $f$  or  $g$  in (3.29)).

Starting with  $A_3^{(i,j)}$ , we use the fact that acting on symmetric functions  $T_i^{-1} = t^{-1}$ , whence

$$A_3^{(i,j)} = -t^{i-1} x_i^{-1} x_j^{-1}, \quad 1 \leq i \leq j \leq n, \quad (3.32)$$

which is certainly Hermitian for all  $i \leq j$ .

Turning to  $A_2^{(i,j)}$ , simplification with (3.3) gives

$$A_2^{(i,j)} = -q T_j \cdots T_{n-1} \omega T_1 \cdots T_{j-1} x_j^{-1} x_i^{-1} T_{i-1} \cdots T_1 \omega^{-1}$$

To proceed further, we must consider the cases  $i = j$ ,  $i < j$  separately. with the results that

$$A_2^{(i,j)} = \begin{cases} -qt^{i-1}T_j \cdots T_{n-1}\omega T_1 \cdots T_i T_{i-1}^{-1} \cdots T_1^{-1}x_1^{-1}x_i^{-1}\omega^{-1} & i < j \\ -qt^{i-1}T_j \cdots T_{n-1}\omega \left( (t^{-1} - 1) \sum_{p=2}^i T_1 \cdots T_{p-1} T_{p-2}^{-1} \cdots T_1^{-1} x_p^{-1} + x_1^{-1} \right) x_1^{-1} \omega^{-1} & i = j \end{cases}$$

Combining these together we obtain after further manipulation

$$\begin{aligned} \sum_{i=1}^n t^{1-i} A_2^{(i,i)} + (1 - t^{-1}) \sum_{1 \leq i < j \leq n} t^{1-i} A_2^{(i,j)} &= q \sum_{i=1}^n T_i \cdots T_{n-1} \omega x_1^{-2} \omega^{-1} \\ &= q^{-1} \sum_{i=1}^n x_i^{-1} \left( (1 - t) \sum_{p=i+1}^n x_p^{-1} + x_i^{-1} \right) \end{aligned} \quad (3.33)$$

which is also Hermitian.

For  $A_1^{(i,j)}$  and  $A_4^{(i,j)}$ , similar considerations hold with the result that

$$\sum_{i=1}^n t^{1-i} A_1^{(i,i)} + (1 - t^{-1}) \sum_{1 \leq i < j \leq n} t^{1-i} A_1^{(i,j)} = \sum_{i=1}^n T_{i-1}^{-1} \cdots T_1^{-1} x_1^{-2} \omega^{-1} \quad (3.34)$$

$$\sum_{i=1}^n t^{1-i} A_4^{(i,i)} + (1 - t^{-1}) \sum_{1 \leq i < j \leq n} t^{1-i} A_4^{(i,j)} = q \sum_{i=1}^n T_i \cdots T_{n-1} \omega x_1^{-2} \quad (3.35)$$

Combining (3.31)–(3.35) together, we finally obtain

$$\begin{aligned} \mathcal{H} = f(x) + \sum_{i=1}^n &\left( t^{-2n+i+1} T_{n-1} \cdots T_1 \omega^{-1} T_{n-1} \cdots T_i \right. \\ &\left. + t^{-n+i} T_{i-1}^{-1} \cdots T_1^{-1} (-(1+a)x_1^{-1} + ax_1^{-2}) \omega^{-1} T_{n-1} \cdots T_i + aq T_i \cdots T_{n-1} \omega x_1^{-2} \right) \end{aligned} \quad (3.36)$$

where  $f(x)$  contains no operators  $T_i^{\pm 1}$ ,  $\omega^{\pm 1}$ , and hence is self-adjoint. To see that this expression is Hermitian with respect to (3.29), denote the 3 terms in the sum as  $\alpha_i$ ,  $\beta_i$  and  $\gamma_i$  respectively. Then

$$\begin{aligned} (\alpha_i + \beta_i)^* &= T_i \cdots T_{n-1} \left\{ t^{-2n+i+1} (\omega^{-1})^* T_1 \cdots T_{n-i} \right. \\ &\quad \left. + t^{-n+i} (-(1+a)x_1^{-1} + ax_1^{-2} \omega^{-1})^* T_1^{-1} \cdots T_{i-1}^{-1} \right\} \\ &= t^{-n+1} T_i \cdots T_{n-1} \left( (1 - x_1^{-1})(1 - ax_1^{-1}) \omega^{-1} \right)^* = \gamma_i, \end{aligned}$$

where  $*$  denotes the adjoint with respect to the inner product (3.29), and we have used the fact that

$$\left( (1 - x_1^{-1})(1 - ax_1^{-1}) \omega^{-1} \right)^* = aqt^{n-1} \omega x_1^{-2}$$

This latter equation follows from the definition (3.2) of  $\omega$  and the adjoint formulas

$$(T_i^{-1})^* = T_i^{-1}, \quad s_i^* = \frac{tx_i - x_{i+1}}{x_i - tx_{i+1}} s_i, \quad \tau_1^* = q^{-1} t^{-n+1} \prod_{l=2}^n \frac{(x_1 - tqx_l)}{(tx_1 - qx_l)} (1 - x_1)(1 - ax_1) \tau_1^{-1}$$

obtainable directly from the definitions of  $T_i, s_i, \tau_1$  and (3.29).

Since  $\mathcal{H}$  is Hermitian with respect to the inner product (3.29) and  $\{U_\kappa^{(a)}\}$  are eigenfunctions of  $\mathcal{H}$  with distinct eigenvalues, it follows immediately that  $\{U_\kappa^{(a)}\}$  are orthogonal with respect to (3.29). Also, by replacing  $q$  by  $q^{-1}$  and using the results (2.23) and (2.34), we see from this result that  $\{V_\kappa^{(a)}\}$  are orthogonal with respect to the inner product

$$\langle f|g \rangle^{(V)} := \int_{[1,\infty]^n} f(x)g(x) d_q \mu^{(V)}(x), \quad d_q \mu^{(V)}(x) := \Delta_q^{(k)}(x) \prod_{l=1}^n w_V(x_l; q) d_q x_l. \quad (3.37)$$

## 4 Special properties and normalization

### 4.1 Special properties

One can check from the generating function (2.6) that the 1-variable polynomials  $U_n$  satisfy the formulas

$$\frac{\partial}{\partial_q x} U_n^{(a)}(x; q) = [n]_q U_{n-1}^{(a)}(x; q), \quad [n]_q := \frac{1 - q^n}{1 - q} \quad (4.1)$$

$$U_n^{(a/q)}(x; q) = U_n^{(a)}(x; q) - q^{-1}a(1 - q^n)U_{n-1}^{(a)}(x; q) \quad (4.2)$$

$$x U_n^{(a)}(x; q) = U_{n+1}^{(a)}(x; q) + (1 + a)q^n U_n^{(a)}(x; q) - aq^{n-1}(1 - q^n)U_{n-1}^{(a)}(x; q) \quad (4.3)$$

$$U_n^{(a)}(1; q) = q^{n(n-1)/2} (-a)^n \quad (4.4)$$

$$U_n^{(a)}(a; q) = q^{n(n-1)/2} (-1)^n \quad (4.5)$$

$$U_n^{(0)}(y; q) = y^n \left(\frac{1}{y}; q\right)_n \quad (4.6)$$

All these formulas have multivariable generalizations. To present these generalizations further notions from the theory of Macdonald polynomials are required.

First we must introduce multivariable analogues of the  $q$ -binomial coefficients  $\binom{\lambda}{\nu}_{q,t}$  associated with the symmetric Macdonald polynomials, which have been the subject of a number of recent studies [15, 22, 17, 24]. Lassalle's [17] starting definition takes the form

$$P_\mu(x; q, t) \prod_{i=1}^n \frac{1}{(x_i; q)_\infty} = \sum_{\lambda} \binom{\lambda}{\mu}_{q,t} t^{b(\lambda) - b(\mu)} \frac{h'_\mu}{h'_\lambda} P_\lambda(x; q, t). \quad (4.7)$$

Of particular importance is  $\binom{\lambda}{\nu}_{q,t}$  when  $\lambda$  and  $\mu$  differ by a single node. Thus, for a given partition  $\lambda$ , let  $\lambda_{(p)}$  (resp.  $\lambda^{(p)}$ ) denote the partition  $\lambda$  with a node removed from (resp. added to) the  $p$ th row, provided the resulting diagram still represents a valid partition. The corresponding binomial coefficients appear in the following Macdonald polynomial identities,

$$E_0 P_\lambda(x; q, t) = \sum_i \binom{\lambda}{\lambda_{(i)}}_{q,t} \frac{P_\lambda(t^{\bar{\delta}}; q, t)}{P_{\lambda_{(i)}}(t^{\bar{\delta}}; q, t)} P_{\lambda_{(i)}}(x; q, t) \quad (4.8)$$

$$e_1(x) P_\lambda(x; q, t) = (1 - q) \sum_i t^{i-1} \binom{\lambda^{(i)}}{\lambda}_{q,t} \frac{h'_\lambda}{h'_{\lambda^{(i)}}} P_{\lambda^{(i)}}(x; q, t) \quad (4.9)$$

where  $t^{\bar{\delta}} := (1, t, t^2, \dots, t^{n-1})$ ,  $e_1(x) := \sum_i x_i$ , and the summations are over those  $i$  such that  $\lambda_{(i)}$ , (resp.  $\lambda^{(i)}$ ) is a valid partition. Furthermore, these particular binomial coefficients have the explicit evaluation [17, Theorem 4]

$$\binom{\lambda}{\lambda_{(p)}}_{q,t} = t^{1-p} \frac{1 - q^{\lambda_p} t^{\ell(\lambda) - p}}{1 - q} \prod_{i=1}^{p-1} \frac{1 - q^{\lambda_i - \lambda_p} t^{p+1-i}}{1 - q^{\lambda_i - \lambda_p} t^{p-i}} \prod_{i=p+1}^{\ell(\lambda)} \frac{1 - q^{\lambda_p - \lambda_i} t^{i-p-1}}{1 - q^{\lambda_p - \lambda_i} t^{i-p}} \quad (4.10)$$

where  $\ell(\lambda)$  denotes the *length* of  $\lambda$ . As these are the only class of binomial coefficients which occur in our generalizations, this result means that the terms in our expressions can be computed explicitly.

Also required is the Pieri formula [21, eq.(6.24)(iv)]

$$e_r(x) P_\mu(x; q, t) = \sum_{\lambda} \psi'_{\lambda/\mu} P_\lambda(x; q, t) \quad (4.11)$$

$\lambda/\mu$  a vertical  $r$ -strip

where  $e_r(x)$  denotes the  $r$ th elementary symmetric function in the variables  $x_1, \dots, x_n$  and

$$\psi'_{\lambda/\mu} := \prod_{(i,j) \in C_{\lambda/\mu} - R_{\lambda/\mu}} \frac{1 - q^{\lambda_i - j} t^{\lambda'_j - i + 1}}{1 - q^{\lambda_i - j + 1} t^{\lambda'_j - i}} \frac{1 - q^{\mu_i - j + 1} t^{\mu'_j - i}}{1 - q^{\mu_i - j} t^{\mu'_j - i + 1}} \quad (4.12)$$

( $C_{\lambda/\mu}$  ( $R_{\lambda/\mu}$ ) denotes the union of columns (rows) which intersect  $\lambda - \mu$ ).

The following result generalizes the first two properties,

**Proposition 4.1** *We have*

$$E_0 U_{\lambda}^{(a)} = \sum_i \binom{\lambda}{\lambda_{(i)}}_{q,t} \frac{P_{\lambda}(t^{\bar{\delta}}; q, t)}{P_{\lambda_{(i)}}(t^{\bar{\delta}}; q, t)} U_{\lambda_{(i)}}^{(a)} \quad (4.13)$$

$$U_{\lambda}^{(a/q)} = \sum_{r=0}^n \left( \frac{-a}{q} \right)^r \sum_{\substack{\mu \\ \lambda/\mu \text{ a vertical } r\text{-strip}}} \psi'_{\lambda/\mu} \frac{f_{\mu}}{f_{\lambda}} U_{\mu}^{(a)} \quad (4.14)$$

where  $f_{\lambda} := t^{b(\lambda)} / (h'_{\lambda}(q, t) P_{\lambda}(t^{\bar{\delta}}; q, t))$ .

**Proof.** The proof of (4.13) follows a similar calculation in the Hermite case [4, Prop. 3.4], while that of (4.14) follows a similar result for Laguerre polynomials [4, Prop. 4.8].  $\square$

**Proposition 4.2** *We have*

$$\begin{aligned} e_1 U_{\lambda}^{(a)} &= (1 + a) e(\lambda) U_{\lambda}^{(a)} + (1 - q) \sum_i t^{i-1} \binom{\lambda^{(i)}}{\lambda}_{q,t} \frac{h'_{\lambda}}{h'_{\lambda_{(i)}}} U_{\lambda_{(i)}}^{(a)} \\ &\quad - a(1 - q) \sum_i q^{\lambda_i - 1} t^{n-i} \binom{\lambda}{\lambda_{(i)}}_{q,t} \frac{P_{\lambda}(t^{\bar{\delta}}; q, t)}{P_{\lambda_{(i)}}(t^{\bar{\delta}}; q, t)} U_{\lambda_{(i)}}^{(a)} \end{aligned}$$

where  $e(\lambda) = \sum_{i=1}^n q^{\lambda_i} t^{n-i}$ .

**Proof.** The proof follows a similar calculation done in [4] for the multivariable Hermite polynomials, using the generating function. The product rule for  $q$ -derivatives complicates matters slightly, but we get around that as follows: define an operator

$$B := \sum_{i=1}^n (1 - x_i)(1 - ax_i) A_i(x; t) \frac{\partial}{\partial_q x_i}$$

which has the important properties

$$\begin{aligned} B^{(x)} \left( \prod_{i=1}^n \rho_a(x_i) {}_0\mathcal{F}_0(x; y; q, t) \right) &= \left( B^{(x)} \prod_{i=1}^n \rho_a(x_i) \right) {}_0\mathcal{F}_0(x; y; q, t) \\ &\quad + \prod_{i=1}^n \rho_a(x_i) \left( E_0^{(x)} {}_0\mathcal{F}_0(x; y; q, t) \right) \end{aligned} \quad (4.15)$$

$$B^{(x)} \prod_{i=1}^n \rho_a(x_i) = \left( \frac{at^{n-1}}{1-q} e_1(x) - \frac{(1+a)[n]_t}{1-q} \right) \prod_{i=1}^n \rho_a(x_i) \quad (4.16)$$

where (4.15) follows from the product rule for  $q$ -differentiation and (4.16) follows directly from computing  $\partial/\partial_q x_i(\rho_a(x_i))$  and using the identities [15]

$$\sum_{i=1}^n A_i(x; t) = [n]_t, \quad \sum_{i=1}^n x_i A_i(x; t) = t^{n-1} e_1(x).$$

Returning to the generating function (2.32) and using the property (3.20), we have (with  $f_\lambda$  as given in Proposition 4.1)

$$\begin{aligned}
\sum_{\lambda} f_{\lambda} e_1(y) U_{\lambda}^{(a)}(y) P_{\lambda}(x; q, t) &= (1-q) \prod_{i=1}^n \rho_a(x_i) \left\{ E_0^{(x)} {}_0\mathcal{F}_0(x; y; q, t) \right\} \\
&= \prod_{i=1}^n \rho_a(x_i) \left\{ (1-q) B^{(x)} - a t^{n-1} e_1(x) + (1+a) [n]_t \right\} {}_0\mathcal{F}_0(x; y; q, t) \\
&= \sum_{\mu} f_{\mu} U_{\mu}^{(a)}(y) \left\{ (1-q) E_0^{(x)} + (1+a) ([n]_t - (1-q) E_1^{(x)}) + \right. \\
&\quad \left. a(1-q) E_2^{(x)} - a t^{n-1} e_1(x) \right\} P_{\mu}(x; q, t)
\end{aligned}$$

where, in the second line we have used (4.15), (4.16) and in the third line, we have used the fact that  $B := E_0 - (1+a)E_1 + aE_2$ . We can now compute the action of the above operators on  $P_{\mu}(x; q, t)$  using (4.8), (4.9) and the relation

$$E_2^{(x)} = \frac{t^{n-1}}{1-q} e_1(x) - \frac{1}{1-q} [E_1^{(x)}, e_1(x)]$$

and then, in the resulting equation, compare coefficients of  $P_{\lambda}(x; q, t)$ , to give us the final result.  $\square$

For the generalization of (4.4) and (4.5) we have

**Proposition 4.3**

$$U_{\lambda}^{(a)}(t^{\bar{\delta}}; q, t) = (-a)^{|\lambda|} q^{b(\lambda')} t^{-b(\lambda)} P_{\lambda}(t^{\bar{\delta}}; q, t) \quad U_{\lambda}^{(a)}(at^{\bar{\delta}}; q, t) = (-1)^{|\lambda|} q^{b(\lambda')} t^{-b(\lambda)} P_{\lambda}(t^{\bar{\delta}}; q, t)$$

**Proof.** The functions  ${}_0\mathcal{F}_0(x; y; q, t)$  and  ${}_0\psi_0(x; y; q, t)$  obey generalizations of (2.4) and (2.5) in the case  $y_i = ct^{i-1}$ , (see [15, 19])

$${}_0\mathcal{F}_0(x; ct^{\bar{\delta}}; q, t) = \prod_{i=1}^n \frac{1}{(cx_i; q)_{\infty}} \quad (4.17)$$

$${}_0\psi_0(x; ct^{\bar{\delta}}; q, t) = \prod_{i=1}^n (cx_i; q)_{\infty}, \quad (4.18)$$

where it is assumed  $|q| < 1$ . Using (4.18) along with the generating function (2.32), we have

$$\sum_{\lambda} \frac{t^{b(\lambda)}}{h'_{\lambda} P_{\lambda}(t^{\bar{\delta}}; q, t)} U_{\lambda}^{(a)}(t^{\bar{\delta}}; q, t) P_{\lambda}(x) = \prod_{i=1}^n (ax_i; q)_{\infty} = {}_0\psi_0(x; at^{\bar{\delta}}; q, t) = \sum_{\lambda} \frac{(-a)^{|\lambda|} q^{b(\lambda')}}{h'_{\lambda}} P_{\lambda}(x; q, t)$$

which yields the first formula on comparison of coefficients of  $P_{\lambda}(x; q, t)$ . The second formula follows similarly.  $\square$

Finally we present the generalization of (4.6).

**Proposition 4.4** Let  $P_{\lambda}^*(x; q, t)$  denote the shifted Macdonald polynomial [22, 16, 24, 17], which is the unique (up to normalization) polynomial of degree  $\leq |\lambda|$  symmetric in the variables  $x_i t^{1-i}$  which vanishes at  $(q_1^{\mu_1}, \dots, q_n^{\mu_n})$  for partitions  $\mu \neq \lambda$ ,  $|\mu| \leq |\lambda|$ . With the normalization such that the leading term of  $P_{\lambda}^*(xt^{-\bar{\delta}}; q, t)$  is  $P_{\lambda}(x; q, t)$  we have

$$U_{\lambda}^{(0)}(y; q, t) = t^{(n-1)|\lambda|} P_{\lambda}^*(yt^{\bar{\delta}-n+1}; q, t).$$

**Proof.** Replace  $x$  by  $t^{-(n-1)}x$  in (4.7), multiply both sides by

$$(-1)^{|\mu|} t^{(n-1)|\mu|} q^{b(\mu')} \frac{P_\mu(y; q, t)}{h'_\mu(q, t) P_\mu(t^{\bar{\delta}}; q, t)}$$

and sum over  $\mu$  using (1.2) to obtain

$$\begin{aligned} {}_0\psi_0(x; y; q, t) & \prod_{i=1}^n \frac{1}{(t^{-(n-1)}x_i; q)_\infty} \\ & = \sum_\lambda \frac{t^{-(n-1)|\lambda|} t^{b(\lambda)}}{h'_\lambda(q, t)} P_\lambda(x; q, t) \sum_\mu \binom{\lambda}{\mu}_{q, t} \frac{q^{b(\mu')} t^{(n-1)|\mu|}}{t^{b(\mu)}} (-1)^{|\mu|} \frac{P_\mu(y; q, t)}{P_\mu(t^{\bar{\delta}}; q, t)}. \end{aligned}$$

Comparing this with the generating function (2.33) with  $a = 0$  gives

$$(-1)^{|\lambda|} q^{b(\lambda')} \frac{V_\lambda^{(0)}(y; q, t)}{P_\lambda(t^{\bar{\delta}}; q, t)} = t^{b(\lambda)} t^{-(n-1)|\lambda|} \sum_\mu \binom{\lambda}{\mu}_{q, t} \frac{q^{b(\mu')}}{t^{b(\mu)}} (-1)^{|\mu|} t^{(n-1)|\mu|} \frac{P_\mu(y; q, t)}{P_\mu(t^{\bar{\delta}}; q, t)}. \quad (4.19)$$

But the shifted Macdonald polynomials have the property that [22, 17]

$$\frac{P_\lambda^*(y; q^{-1}, t^{-1})}{P_\lambda^*(0; q^{-1}, t^{-1})} = \sum_\mu \binom{\lambda}{\mu}_{q, t} \frac{q^{b(\mu')}}{t^{b(\mu)}} (-1)^{|\mu|} \frac{P_\mu(y t^{\bar{\delta}}; q, t)}{P_\mu(t^{\bar{\delta}}; q, t)}. \quad (4.20)$$

Furthermore [22],  $P_\lambda^*(0; q^{-1}, t^{-1}) = (-1)^{|\lambda|} q^{b(\lambda')} t^{-b(\lambda)} P_\lambda(t^{-\bar{\delta}}; q, t)$ , so substituting (4.20) in (4.19) we see that

$$V_\lambda^{(0)}(y; q, t) = t^{-(n-1)|\lambda|} P_\lambda^*(y t^{-\bar{\delta}+n-1}; q^{-1}, t^{-1}).$$

The stated equation follows from this by replacing  $q$  and  $t$  by their reciprocals.  $\square$

We remark that it is possible to use (4.19) to deduce an operator formula relating  $V_\lambda^{(0)}(y; q, t)$  and  $\binom{\lambda}{\mu}_{q, t}$ , which is the symmetric  $q$ -analogue of a formula given in ref. [5, proof of Prop. 3.23]. For this we require a  $q$ -analogue of the Dunkl pairing [9].

**Definition 4.5** With  $D_i$  given by (3.6), and  $p$  and  $q$  symmetric polynomials of  $n$  variables, define the pairing

$$[p, q] := p(D)q \Big|_{x=0}, \quad (4.21)$$

where  $p(D)$  is the operator obtained from  $p(x)$  by replacing each  $x_i$  by  $D_i$ .

For the Dunkl pairing defined by (4.21) with the  $D_i$  replaced by the  $A$  type Dunkl operators

$$d_i := \frac{\partial}{\partial x_i} - \sum_{j \neq i} \frac{1 - s_{ij}}{x_i - x_j},$$

the Jack polynomials form an orthogonal set. For the pairing (3.6) as written, the Macdonald polynomials form an orthogonal set.

**Proposition 4.6** We have

$$[P_\kappa(x; q, t), P_\sigma(x; q, t)] = t^{-b(\kappa)} h'_\kappa(q, t) P_\kappa(t^{\bar{\delta}}; q, t) \delta_{\kappa, \sigma}.$$

**Proof.** Take  $f = P_\kappa(x; q, t)$  in (3.8) and set  $x = 0$ . This gives

$$P_\kappa(D^{(x)}; q, t) {}_0\mathcal{F}_0(x; y; q, t) \Big|_{x=0} = P_\kappa(y; q, t).$$

Equating coefficients of  $P_\sigma(y; q, t)$  on both sides gives the stated result.  $\square$

Acting on both sides of (4.19) with  $P_\mu(D^{(y)}; q, t)$  and setting  $y = 0$ , Proposition 4.6 immediately implies the sought operator formula.



**Proposition 4.7** *We have*

$$(-1)^{|\lambda|+|\mu|} \frac{q^{b(\lambda')-b(\mu')} t^{(n-1)(|\lambda|-|\mu|)} t^{2b(\mu)-b(\lambda)}}{P_\lambda(t^\delta; q, t) h'_\mu(q, t)} P_\mu(D^{(y)}; q, t) V_\lambda^{(0)}(y; q, t) \Big|_{y=0} = \binom{\lambda}{\mu}_{q, t}.$$

## 4.2 Normalization integral for $\lambda = 0$

For  $f = g = 1$ , the inner product (3.29) reads

$$\int_{[a, 1]^n} \prod_{l=1}^n \frac{E_q(-qx_l) E_q(-\frac{qx_l}{a})}{E_q(-q) E_q(-a) E_q(-\frac{q}{a})} \prod_{p=-(k-1)}^k \prod_{1 \leq i < j \leq n} (x_i - q^p x_j) d_q x_1 \cdots d_q x_n := \mathcal{N}_0^{(U)}(a; q, t) \quad (4.22)$$

To evaluate  $\mathcal{N}_0^{(U)}$  we use a modification of the method used by Aomoto [2] to evaluate the  $q$ -Selberg integral. There are two main steps. The first is to specify the dependence on  $a$ . This we do by relating  $\mathcal{N}_0^{(U)}(aq; q, t)$  to  $\mathcal{N}_0^{(U)}(a; q, t)$ . In fact we will show that

$$\mathcal{N}_0^{(U)}(aq; q, t) = t^{n(n-1)/2} \mathcal{N}_0^{(U)}(a; q, t), \quad (4.23)$$

which implies

$$\mathcal{N}_0^{(U)}(a; q, t) = a^{kn(n-1)/2} g(q, t). \quad (4.24)$$

The second step is to observe that the unknown function  $g(q, t)$  can be written as a limit:

$$g(q, t) = \lim_{a \rightarrow 0} a^{-kn(n-1)/2} \mathcal{N}_0^{(U)}(a; q, t), \quad (4.25)$$

and to evaluate the limit. Inspection of the summand of the multiple sum defining  $\mathcal{N}_0^{(U)}(a; q, t)$  for  $a \rightarrow 0$  shows that the limiting behaviour is determined by the term  $x = t^\delta$ . Hence

$$\mathcal{N}_0^{(U)}(a; q, t) \sim (1-q)^n t^{\sum_{j=1}^n (j-1)} \prod_{i=1}^n w_U(x_i; q) \Delta_q^{(k)}(x) \Big|_{x=t^\delta}, \quad (4.26)$$

where the notation used in (3.29) has been reintroduced. Now, for  $a \rightarrow 0$  and  $t = q^k$ ,

$$w_U(t^{j-1}q) \sim q^{-\frac{1}{2}k(j-1)(k(j-1)+1)} \frac{(-a)^{k(j-1)}}{(q; q)_{k(j-1)}}$$

while

$$\Delta_q^{(k)}(t^\delta) = t^{2k \sum_{j=1}^{n-2} j(n-1-j)} \frac{(q; q)_{kn} \left( \prod_{l=1}^{n-1} (q; q)_{k(n-l)} \right)^2}{(q; q)_k^n}.$$

Making use of the summation  $\sum_{l=1}^{n-1} l^2 = \frac{1}{3}n^3 - \frac{1}{2}n^2 + \frac{1}{6}n$ , substituting these expressions in (4.26), then substituting the resulting expression in (4.25) gives the evaluation

$$\mathcal{N}_0^{(U)}(a; q, t) = (1-q)^n (-a)^{kn(n-1)/2} t^{k \binom{n}{3} - \frac{k-1}{2} \binom{n}{2}} \prod_{l=1}^n \frac{(q; q)_{kl}}{(q; q)_k}. \quad (4.27)$$

We remark that with  $a = -1$ , in the limit  $q \rightarrow 1$  (4.27) gives a limiting case of the Selberg integral known as the Mehta integral,

$$\prod_{l=1}^n \int_{-\infty}^{\infty} dt_l e^{-\frac{1}{2}t_l^2} \prod_{1 \leq j < k \leq n} |t_k - t_j|^{2k} = (2\pi)^{n/2} \prod_{j=0}^{n-1} \frac{\Gamma(1 + (j+1)k)}{\Gamma(1+k)} \quad (4.28)$$

(this is the integral (2.30) with  $y_j^2 = \frac{1}{2}t_j^2$  and  $1/\alpha = k$ ). Furthermore, by replacing  $q$  by  $q^{-1}$  and using the result (2.23) we deduce from (4.22) that for  $t = q^k$

$$\int_{[1,\infty]^n} d_q \mu^{(V)}(x) =: \mathcal{N}_0^{(V)}(a; q, t) = (1-q)^n a^{kn(n-1)/2} t^{-2k\binom{n}{3}-k\binom{n}{2}} \prod_{l=1}^n \frac{(q; q)_{kl}}{(q; q)_k}. \quad (4.29)$$

It remains to establish (4.23). Now, it follows immediately from (2.8) and (4.22) that

$$\mathcal{N}_0^{(U)}(aq; q, t) = \int_{[a,1]^n} \prod_{j=1}^n (x_j - a) d_q \mu^{(U)}(x). \quad (4.30)$$

But in Appendix A we will show that

$$\prod_{j=1}^n (x_j - a) = \sum_{r=0}^n t^{\binom{n-r}{2}} U_{(1^r)}^{(a)}(x) \quad (4.31)$$

Substituting (4.31) in (4.30), noting from (2.35) that  $U_{(0)} = 1$ , and using the orthogonality of  $\{U_\kappa^{(a)}\}$  with respect to (3.29) we obtain (4.23).

J. Stokman has pointed out to us that our evaluation (4.27) is a special case of a result of Evans [11], in which the ratio of  $q$ -exponentials  $E_q$  in (4.22) is replaced by the weight function (5.1) below, and the integration domain  $[a, 1]^n$  is replaced by  $[-d, c]^n$ .

### 4.3 Normalization integral for general $\lambda$

Suppose we have a set of polynomials  $\{U_\lambda(x)\}$ , orthogonal with respect to some inner product  $\langle \cdot, \cdot \rangle$  in which  $e_1$  is self-adjoint. Moreover, suppose we have a “first” Pieri formula of the form

$$e_1(x) U_\mu(x) = \sum_i a_i(\mu) U_{\mu^{(i)}}(x) + c(\mu) U_\mu(x) + \sum_i b_i(\mu) U_{\mu_{(i)}}(x)$$

and that there exists a  $p$ , such that  $\mu^{(p)} = \lambda$ . Then using the orthogonality of the polynomials  $\{U_\lambda(x)\}$ , we have

$$\begin{aligned} \langle U_\lambda, U_\lambda \rangle &= \frac{1}{a_p(\mu)} \langle e_1 U_\mu, U_\lambda \rangle = \frac{1}{a_p(\mu)} \langle U_\mu, e_1 U_\lambda \rangle \\ &= \frac{b_p(\lambda)}{a_p(\mu)} \langle U_\mu, U_\mu \rangle = \frac{b_p(\lambda)}{a_p(\lambda_{(p)})} \langle U_{\lambda_{(p)}}, U_{\lambda_{(p)}} \rangle \end{aligned} \quad (4.32)$$

Now in general, there is no unique way to go from a partition  $\nu$  to a partition  $\lambda \supset \nu$  by adding one node at a time. Indeed, there are  $f^{\lambda/\nu}$  ways of doing this, where  $f^{\lambda/\nu}$  is the number of standard tableaux of shape  $\lambda/\nu$ . Thus, the relation (4.32) can only be iterated *provided*

$$\frac{b_p(\lambda)}{a_p(\lambda_{(p)})} = \frac{g(\lambda)}{g(\lambda_{(p)})} \quad (4.33)$$

for some function  $g$ , in which case there is no inconsistency i.e. all sequences going from  $\lambda$  to  $\nu$  by removing one node at a time will yield the same result. If this condition is satisfied, then clearly

$$\frac{\langle U_\lambda, U_\lambda \rangle}{\langle U_\nu, U_\nu \rangle} = \frac{g(\lambda)}{g(\nu)}$$

In the case of the Al-Salam&Carlitz polynomials  $U_\lambda^{(a)}(x)$ , it follows from Proposition 4.2 that (4.33) is indeed satisfied with

$$g(\lambda) = (-at^{n-1})^{|\lambda|} q^{b(\lambda')} t^{-2b(\lambda)} h'_\lambda P_\lambda(t^{\bar{\delta}}; q, t).$$

Hence

$$\langle U_\lambda, U_\lambda \rangle^{(U)} =: \mathcal{N}_\lambda^{(U)}(a; q, t) = (-at^{n-1})^{|\lambda|} q^{b(\lambda')} t^{-2b(\lambda)} h'_\lambda P_\lambda(t^{\bar{\delta}}; q, t) \mathcal{N}_0^{(U)}(a; q, t). \quad (4.34)$$

By replacing  $q$  by  $q^{-1}$  and using (2.23), (1.3) and (1.4) we obtain the normalization formula for  $\{V_\kappa^{(a)}\}$  with respect to the inner product (3.37),

$$\langle V_\lambda, V_\lambda \rangle^{(V)} =: \mathcal{N}_\lambda^{(V)}(a; q, t) = (aq^{-1}t^{-2(n-1)})^{|\lambda|} q^{-2b(\lambda')} t^{b(\lambda)} h'_\lambda P_\lambda(t^{\bar{\delta}}; q, t) \mathcal{N}_0^{(V)}(a; q, t). \quad (4.35)$$

#### 4.4 Integral representations

The formulas (4.34) and (4.35), together with the orthogonality property of  $\{U_\kappa^{(a)}\}$  and  $\{V_\kappa^{(a)}\}$  allow  $U_\kappa^{(a)}$  and  $V_\kappa^{(a)}$  to be expressed in terms of certain  $q$ -integrals in which  ${}_0\mathcal{F}_0^{(\alpha)}(x; y)$  and  ${}_0\psi_0(x; y; q, t)$  occur as kernels. Here the convergence properties of  ${}_0\mathcal{F}_0(x; y; q, t)$  and  ${}_0\psi_0(x; y; q, t)$  are relevant. Suppose  $0 < q, t < 1$ . A formula of Macdonald [21, (7.13')] gives that  $P_\kappa(y; q, t)$  then has positive coefficients when expressed as a series in monomial symmetric functions, so we have that  $|P_\kappa(y; q, t)| \leq P_\kappa(|y|; q, t) \leq c^{|\kappa|} P_\kappa(t^{\bar{\delta}}; q, t)$  for some  $c > \max(t^{-(n-1)}|y_j|)_{j=1, \dots, n}$ . Thus

$$|{}_0\mathcal{F}_0(y; x; q, t)| \leq {}_0\mathcal{F}_0(ct^{\bar{\delta}}; |x|; q, t) = \prod_{i=1}^n \frac{1}{(c|x_i|; q)_\infty}. \quad (4.36)$$

Similarly we have

$$|{}_0\psi_0(y; x; q, t)| \leq {}_0\psi_0(-ct^{\bar{\delta}}; |x|; q, t) = \prod_{i=1}^n (-c|x_i|; q)_\infty. \quad (4.37)$$

The results to be presented are the  $q$ -analogues of Proposition 3.8 and Corollaries 3.1 and 3.2 of ref. [4].

**Proposition 4.8** *For  $|y|$  and  $|z|$  small enough such that all quantities converge,*

$$\begin{aligned} & \frac{1}{(1-q)^n} \int_{[a,1]^n} {}_0\mathcal{F}_0(y; x; q, t) {}_0\mathcal{F}_0(z; x; q, t) d_q \mu^{(U)}(x) \\ &= \mathcal{N}_0^{(U)}(a; q, t) \prod_{l=1}^n \frac{1}{\rho_a(y_l; q) \rho_a(z_l; q)} {}_0\psi_0(y; at^{n-1}z; q, t), \end{aligned} \quad (4.38)$$

$$\begin{aligned} & \frac{1}{(1-q)^n} \int_{[1,\infty]^n} {}_0\psi_0(y; x; q, t) {}_0\psi_0(z; x; q, t) d_q \mu^{(V)}(x) \\ &= \mathcal{N}_0^{(V)}(a; q, t) \prod_{l=1}^n \rho_a(t^{-(n-1)}y_l; q) \rho_a(t^{-(n-1)}z_l; q) {}_0\mathcal{F}_0(y; aq^{-1}t^{-2(n-1)}z; q, t). \end{aligned} \quad (4.39)$$

**Proof.** The bounds (4.36) and (4.37), together with the definitions (3.29) and (3.37), show that for  $|y|$  and  $|z|$  small enough the integrals converge. To verify (4.38), multiply both sides by  $\prod_{l=1}^n \rho_a(y_l; q) \rho_a(z_l; q)$  and substitute for  $\prod_{l=1}^n \rho_a(y_l; q) {}_0\mathcal{F}_0(y; x; q, t)$  and  $\prod_{l=1}^n \rho_a(z_l; q) {}_0\mathcal{F}_0(z; x; q, t)$  using the generating function (2.32). Now integrate term-by-term using the orthogonality of  $\{U_\kappa^{(a)}\}$  with respect to the inner product (3.29) and the normalization integral (4.34), and identify the resulting series according to the definition (1.2). The equation (4.39) follows by replacing  $q, t$  by  $q^{-1}, t^{-1}$  in (4.38).  $\square$

**Corollary 4.9** *We have*

$$\begin{aligned} & \frac{1}{(1-q)^n} \int_{[a,1]^n} {}_0\mathcal{F}_0(y; x; q, t) U_\kappa^{(a)}(x; q, t) d_q \mu^{(U)}(x) \\ &= \mathcal{N}_0^{(U)}(a; q, t) (-at^{n-1})^{|\kappa|} q^{b(\kappa')} t^{-b(\kappa)} \prod_{l=1}^n \frac{1}{\rho_a(y_l; q)} P_\kappa(y; q, t) \end{aligned} \quad (4.40)$$

$$\begin{aligned} & \frac{1}{(1-q)^n} \int_{[1,\infty)^n} {}_0\psi_0(y; x; q, t) V_\kappa^{(a)}(x; q, t) d_q \mu^{(V)}(x) \\ &= \mathcal{N}_0^{(V)}(a; q, t) (-aq^{-1}t^{-2(n-1)})^{|\kappa|} q^{-b(\kappa')} t^{b(\kappa)} \prod_{l=1}^n \rho_a(t^{-(n-1)}y_l; q) P_\kappa(y; q, t) \end{aligned} \quad (4.41)$$

$$\begin{aligned} & \frac{1}{(1-q)^n} \int_{[a,1]^n} {}_0\mathcal{F}_0(z; x; q, t) P_\kappa(x; q, t) d_q \mu^{(U)}(x) \\ &= \mathcal{N}_0^{(U)}(a; q, t) (-t^{n-1})^{|\kappa|} q^{b(\kappa')} t^{-b(\kappa)} \prod_{l=1}^n \frac{1}{\rho_a(z_l; q)} V_\kappa^{(a)}(az; q, t) \end{aligned} \quad (4.42)$$

$$\begin{aligned} & \frac{1}{(1-q)^n} \int_{[1,\infty)^n} {}_0\psi_0(y; x; q, t) P_\kappa(x; q, t) d_q \mu^{(V)}(x) \\ &= \mathcal{N}_0^{(V)}(a; q, t) (-aq^{-1}t^{-2(n-1)})^{|\kappa|} q^{-b(\kappa')} t^{b(\kappa)} \prod_{l=1}^n \rho_a(t^{-(n-1)}z_l; q) U_\kappa^{(a)}(az; q, t) \end{aligned} \quad (4.43)$$

**Proof.** These formulas follow from (4.38) and (4.39) by using the generating functions (2.32) and (2.33) in an analogous way to the proof of Corollaries 3.1 and 3.2 of [4]. We remark that (4.40) and (4.41), and (4.42) and (4.43), are equivalent under the mapping  $q \mapsto q^{-1}$ .  $\square$

## 5 Relationship to the big $q$ -Jacobi polynomials

It has been pointed out to us by J. Stokman that the Al-Salam&Carlitz polynomials  $U_\lambda^{(a)}$  introduced herein can be viewed as the special case  $a = b = 0$ ,  $c = 1$  of his recently introduced [26] multivariable big  $q$ -Jacobi polynomials  $P_\lambda^B(\cdot; a, b, c, d; q, t)$ , with the parameter  $d$  equal to  $-a$  in  $U_\lambda^{(a)}$ . This can be seen from the fact [26, eq. (5.18)] that for  $t = q^k$ ,  $k \in \mathbb{Z}^+$ , the polynomials  $P_\lambda^B$  are orthogonal with respect to the Jackson integral inner product (3.29) modified so that  $[a, 1]^n$  is replaced by  $[-d, c]^n$  and  $w_U(x; q)$  is replaced by

$$w_B(x; q) := \frac{(qx/c; q)_\infty (-qx/d; q)_\infty}{(qax/c; q)_\infty (-qbx/d; q)_\infty}. \quad (5.1)$$

The weight function  $w_B$  reduces to  $w_U$  for the specified values of the parameters.

An important implication of this identification concerns the orthogonality for  $t \neq q^k$ . In [26] it was proved that the  $P_\lambda^B$  are orthogonal with respect to a sum of particular iterated Jackson integrals [26, eq. (5.2)], which reduces to the modification of the Jackson integral (3.29) specified above when  $t = q^k$ . For the choice of parameters specified above in this more general inner product, the polynomials  $U_\lambda^{(a)}$  will thus form an orthogonal set for general  $t \in (0, 1)$ .

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## Appendix A

In this appendix, we derive the expansion for the product  $\prod_{i=1}^n (x_i - a)$  in terms of Al-Salam–Carlitz polynomials  $U_\lambda^{(a)}(x)$ . There seems to be no direct way to do this utilizing the generating function (2.32), so we proceed indirectly by using the characterization of  $U_\lambda^{(a)}(x)$  (resp.  $P_\lambda(x; q, t)$ ) as an eigenfunction of  $\mathcal{H}$  (resp.  $\widetilde{M}_1$ ) in the particular case  $\lambda = (1^p)$ . Since the product

$$\prod_{i=1}^n (x_i - a) = (-a)^n \prod_{i=1}^n (1 - x_i/a) = (-a)^n \sum_{r=0}^n e_r(x) \left( \frac{-1}{a} \right)^r$$

our aim will be to expand the elementary symmetric function  $e_r(x)$  in terms of the  $U_{(1^p)}^{(a)}(x)$ . Specifically, we proceed in three stages: Expand  $U_{(1^p)}^{(a)}$  in terms of  $e_i$ ,  $0 \leq i \leq p$ ; compute the action of  $\widetilde{M}_1$  on  $U_{(1^p)}^{(a)}$  in terms of  $U_{(1^i)}^{(a)}$ ,  $i = p, p-1, p-2$ ; expand  $e_p$  in terms of  $U_{(1^i)}^{(a)}$ ,  $0 \leq i \leq p$ .

### Step 1.

We first compute the action of the operator  $\mathcal{H}$  on the elementary symmetric function  $e_p$  using the following results:

$$\begin{aligned} \widetilde{M}_1 e_p &= \tilde{e}(1^p) e_p, & \tilde{e}(\lambda) &:= \sum_{i=1}^n q^{-\lambda_i} t^{-n+i} \\ E_0 e_p &= [n+1-p]_t e_{p-1} \end{aligned}$$

where  $[n]_t := (1 - t^n)/(1 - t)$ . This latter identity appears in the work of Kaneko [15]. Using these identities and the form of  $\mathcal{H}$  given in Proposition 3.1, it follows that

$$\mathcal{H} e_p = A_1^{(p)} e_p + A_2^{(p)} e_{p-1} + A_3^{(p)} e_{p-2}$$

where

$$\begin{aligned} A_1^{(p)} &= \tilde{e}(1^p) & A_2^{(p)} &= -(1+a)(q^{-1} - 1)t^{-n+p}[n+1-p]_t \\ A_3^{(p)} &= a(q^{-1} - 1)(t-1)t^{-n+p-1}[n+1-p]_t[n+2-p]_t \end{aligned}$$

Suppose we have the expansion

$$U_{(1^p)} = \sum_{i=0}^p a_i^{(p)} e_i \quad a_p^{(p)} = 1$$

If we now use the fact that  $\mathcal{H}U_{(1^p)} = \tilde{e}(1^p)U_{(1^p)}$  and compare coefficients of  $e_i$ , we obtain a set of 3-term recurrence relations for the coefficients  $a_i^{(p)}$ . Indeed, if

$$a_{p-i}^{(p)} = f_i(a) \begin{bmatrix} n-p+i \\ i \end{bmatrix}_t, \quad \begin{bmatrix} n \\ r \end{bmatrix}_t = \frac{(t; t)_n}{(t; t)_r (t; t)_{n-r}}$$

then the polynomials  $f_i(a)$  obey the 3-term relation

$$f_i = -(1+a) f_{i-1} + a(t^{i-1} - 1) f_{i-2} \quad f_0(a) = 1, \quad f_1(a) = -(1+a) \quad (\text{A.1})$$

Hence

$$U_{(1^p)} = \sum_{i=0}^p f_i(a) \left[ \begin{matrix} n-p+i \\ i \end{matrix} \right]_t e_{p-i}. \quad (\text{A.2})$$

By comparison with the 3-term relation for the polynomials  $V_n^{(a)}(x; q)$  (see [13, eq. (5.4)]), we see that in fact  $f_i(a) := q^{i(i-1)/2} V_i^{(a)}(0; q)$ .

### Step 2.

We can now use the expansion (A.2) to calculate the action of  $M_1$  on  $U_{(1^p)}$ . Indeed, we expect

$$M_1 U_{(1^p)} = \gamma_1^{(p)} U_{(1^p)} + \gamma_2^{(p)} U_{(1^{p-1})} + \gamma_3^{(p)} U_{(1^{p-2})} \quad (\text{A.3})$$

If we now insert the expansion (A.2) into this equation and compare coefficients of  $e_{p-i}$ , we find that, due to the 3-term relation (A.1),

$$\begin{aligned} \gamma_1^{(p)} &= e(1^p) & e(\lambda) &:= \sum_{i=1}^n q^{\lambda_i} t^{n-i} \\ \gamma_2^{(p)} &= -(1-q)(1+a)t^{n-p}[n+1-p]_t \\ \gamma_3^{(p)} &= (1-q)(t-1)at^{n-p}[n+1-p]_t[n+2-p]_t \end{aligned}$$

### Step 3.

Finally, we can use (A.3) to obtain the expansion

$$e_p = \sum_{i=0}^p b_i^{(p)} U_{(1^i)}, \quad b_p^{(p)} = 1 \quad (\text{A.4})$$

This is done by using the fact that  $M_1 e_p = e(1^p) e_p$  and following the procedure used in step 1, to obtain a 3-term recurrence relation for the coefficients  $b_i^{(p)}$ . This recurrence relation has the solution

$$b_{p-i}^{(p)} = \tilde{f}_i(a) \left[ \begin{matrix} n-p+i \\ i \end{matrix} \right]_t \quad (\text{A.5})$$

where the polynomials  $\tilde{f}_i(a)$  obey the 3-term relation

$$\tilde{f}_i = (1+a)t^{i-1} \tilde{f}_{i-1} + at^{i-2}(1-t^{i-1})\tilde{f}_{i-2}, \quad \tilde{f}_0 = 1, \quad \tilde{f}_1 = 1+a \quad (\text{A.6})$$

Again, comparing with the 3-term relation for the polynomials  $U_n^{(a)}(x; q)$  given in (4.3), we see that  $\tilde{f}_i(a) = (-1)^i U_i^{(a)}(0; q)$ .

We can now use (A.4), (A.5) to expand

$$\prod_{i=1}^n (1 + x_i z) = \sum_{p=0}^n e_p z^p = \sum_{r=0}^n \left( \sum_{j=r}^n b_r^{(j)} z^j \right) U_{(1^r)} \quad z := -1/a$$

However, the internal summation can be summed exactly, and we claim that

$$\sum_{j=r}^n b_r^{(j)} \left( \frac{-1}{a} \right)^j = (-a)^{-n} t^{\binom{n-r}{2}} \quad (\text{A.7})$$

Using the explicit expression for the coefficients  $b_i^{(p)}$ , this is equivalent to proving (after changing the variables in the summation)

$$\sum_{i=0}^{n-r} \tilde{f}_{n-r-i}(a) \begin{bmatrix} n-r \\ i \end{bmatrix}_t (-a)^i = t^{\binom{n-r}{2}} \quad (\text{A.8})$$

Denote the left hand side of (A.8) by  $S_{n-r}$ . It suffices to show that  $S_m = t^{m-1} S_{m-1}$ . We first note the following identities for  $t$ -binomial coefficients:

$$\begin{bmatrix} m \\ i \end{bmatrix}_t = \begin{bmatrix} m-1 \\ i \end{bmatrix}_t + t^{m-i} \begin{bmatrix} m-1 \\ i-1 \end{bmatrix}_t \quad (\text{A.9})$$

$$t^i \begin{bmatrix} m-1 \\ i \end{bmatrix}_t = \begin{bmatrix} m-1 \\ i \end{bmatrix}_t - (1-t^{m-i}) \begin{bmatrix} m-1 \\ i-1 \end{bmatrix}_t \quad (\text{A.10})$$

Using (A.9), we have

$$\begin{aligned} S_m &= \sum_{i=0}^m \tilde{f}_{m-i} \left( \begin{bmatrix} m-1 \\ i \end{bmatrix}_t + t^{m-i} \begin{bmatrix} m-1 \\ i-1 \end{bmatrix}_t \right) (-a)^i \\ &= \sum_{i=0}^{m-1} \tilde{f}_{m-i} \begin{bmatrix} m-1 \\ i \end{bmatrix}_t (-a)^i + \sum_{i=0}^{m-1} \tilde{f}_{m-i-1} t^{m-i-1} \begin{bmatrix} m-1 \\ i \end{bmatrix}_t (-a)^{i+1} \end{aligned}$$

Now apply the 3-term relation (A.6) to the first term which yields

$$\begin{aligned} S_m &= \sum_{i=0}^{m-1} \left( t^{m-i-1} \tilde{f}_{m-i-1} + at^{m-i-2}(1-t^{m-i-1})\tilde{f}_{m-i-2} \right) \begin{bmatrix} m-1 \\ i \end{bmatrix}_t (-a)^i \\ &= \sum_{i=0}^{m-1} t^{m-i-1} \tilde{f}_{m-i-1} \left( \begin{bmatrix} m-1 \\ i \end{bmatrix}_t - (1-t^{m-i-1}) \begin{bmatrix} m-1 \\ i-1 \end{bmatrix}_t \right) (-a)^i \end{aligned}$$

where to obtain the last line, we have shifted summation variables in the second term of the previous line. Using (A.10) thus gives  $S_m = t^{m-1} S_{m-1}$  as desired.

We thus have the expansion (4.31).

## Appendix B

Here we will show that in the special case  $t = q$ , the polynomials  $U_\kappa^{(a)}$  admit a determinant formula in terms of their one-variable counterparts  $U_j^{(a)}$ . Now, in the case  $t = q$ ,  $\{U_\kappa^{(a)}\}$  form an orthogonal set with respect to the inner product

$$\langle f, g \rangle = \int_{[a,1]^n} \prod_{l=1}^n w_U(x_l; q) \prod_{1 \leq j < k \leq n} (x_j - x_k)(x_j - qx_k) f(x_1, \dots, x_n) g(x_1, \dots, x_n) d_q x_1 \cdots d_q x_n,$$

where  $f$  and  $g$  are assumed symmetric. According to a lemma of Kadell [14], the product  $\prod_{1 \leq j < k \leq n} (x_j - qx_k)$  can be replaced by  $\prod_{1 \leq j < k \leq n} (x_j - x_k)$  provided the integral is multiplied by  $[n]_q! / n!$ . Thus we can write

$$\langle f, g \rangle = \frac{[n]_q!}{n!} \int_{[a,1]^n} \prod_{l=1}^n w_U(x_l; q) \prod_{1 \leq j < k \leq n} (x_j - x_k)^2 f(x_1, \dots, x_n) g(x_1, \dots, x_n) d_q x_1 \cdots d_q x_n, \quad (\text{B.1})$$

and we can characterize the corresponding Al-Salam&Carlitz polynomials as the unique symmetric polynomials with an expansion of the form (2.35) which are orthogonal with respect to (B.1).

Using this characterization, we can easily verify the determinant formula.

**Proposition B.1** *We have*

$$U_{\kappa}^{(a)}(x; q, q) = \frac{\det[U_{\kappa_i+n-i}^{(a)}(x_j; q)]_{i,j=1,\dots,n}}{\prod_{1 \leq i < j \leq n} (x_j - x_i)} \quad (\text{B.2})$$

**Proof.** First note that the ratio (B.2) is indeed a polynomial. The highest degree leading terms are obtained by replacing each  $U_{\kappa_i+n-i}^{(a)}(x_j; q)$  by  $x_j^{\kappa_i+n-i}$ . This gives the Schur polynomial  $s_{\kappa}(x)$ , and verifies the requirement (2.35), since  $P_{\kappa}(x; q, q) = s_{\kappa}(x)$ . Now, substituting (B.2) in (B.1) gives

$$\begin{aligned} \langle U_{\kappa}^{(a)}, U_{\mu}^{(a)} \rangle &= \frac{[n]_q!}{n!} \int_{[a,1]^n} \prod_{l=1}^n w_U(x_l; q) \det[U_{\kappa_i+n-i}^{(a)}(x_j; q)]_{i,j=1,\dots,n} \det[U_{\mu_i+n-i}^{(a)}(x_j; q)]_{i,j=1,\dots,n} \\ &\quad \times d_q x_1 \cdots d_q x_n \\ &= [n]_q! \int_{[a,1]^n} \prod_{l=1}^n w_U(x_l; q) U_{\kappa_l+n-l}^{(a)}(x_l; q) \det[U_{\mu_i+n-i}^{(a)}(x_j; q)]_{i,j=1,\dots,n} d_q x_1 \cdots d_q x_n. \end{aligned}$$

Since  $\{U_j^{(a)}(x; q)\}$  is an orthogonal set with respect to the one-dimensional inner product  $\langle f, g \rangle := \int_a^1 w_U(x; q) f(x) g(x) d_q x$ , integrating row-by-row in the determinant gives

$$\begin{aligned} \langle U_{\kappa}^{(a)}, U_{\mu}^{(a)} \rangle &= [n]_q! \left( \prod_{i=1}^n \langle U_{\kappa_i+n-i}^{(a)}, U_{\mu_i+n-i}^{(a)} \rangle \right) \delta_{\kappa, \mu} \\ &= [n]_q! (1-q)^n (-a)^{|\kappa|+n(n-1)/2} q^{\sum_{i=1}^n (\kappa_i+n-i)(\kappa_i+n-i-1)/2} \prod_{i=1}^n (q; q)_{\kappa_i+n-i} \delta_{\kappa, \mu}, \end{aligned} \quad (\text{B.3})$$

where the final equality follows by using (2.7), thus establishing the orthogonality.  $\square$

We remark that straightforward manipulation shows that (B.3) agrees with the normalization formulas (4.27) and (4.34).

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